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On the Axisymmetric Problem of Elasticity Theory for a Medium with Transverse Isotropy

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1. Introduction

The linear theory of elasticity of anisotropic media has long been the subject of numerous investigations. In particular, the problems of torsion and flexure of beams for various underlying anisotropic stress-strain relations, as well as the plane problem for an orthotropic medium, have been treated extensively in the literature which now contains a wealth of significant solutions of such problems.[1]

The abandonment of the assumption of elastic isotropy in three-dimensional problems, on the other hand, leads to greater analytical complications. Here, apparently, only the case in which the material possesses an axis of elastic symmetry ("transverse isotropy"[2]), as illustrated by crystals of the hexagonal system, has received successful attention. The first investigation of this kind is due to Michell [2] (1900), who obtained the solution corresponding to a transversely isotropic semi-infinite elastic body under arbitrarily prescribed surface tractions on its plane boundary.

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[1] A bibliography of two-dimensional investigations based on anisotropic stress-strain laws, is beyond the scope of the present paper.

[2] This terminology is used by Love [1], p. 160. Numbers in brackets refer to the bibliography at the end of the paper.

Further progress in this connection was made by Lekhnitsky [3] (1940), [4], who stated a stress-function approach to problems characterized by torsionless axisymmetry for the case in which the axis of elastic symmetry coincides with the axis of stress symmetry. Lekhnitsky's solution of the displacement equations of equilibrium in terms of a single stress function satisfying a fourth-order partial differential equation, is a generalization of Love's approach[3] to isotropic rotationally symmetric problems, which in turn may be regarded[4] as a specialization of Galerkin's [6] general solution of the three-dimensional field equations. Lekhnitsky [3] employed his approach in securing several technically important particular solutions.

Subsequently, Elliot [7] (1948) gave a solution of the three-dimensional field equations for a transversely isotropic medium in terms of two stress functions, each satisfying a second-order partial differential equation. This solution, in the special instance of rotational stress symmetry about the axis of elastic symmetry, reduces to that originated by Lekhnitsky [3]. Elliot's approach, which is evidently incomplete in the absence of axisymmetry of the stress field, has been applied by him [7], [8] and by Shield [9] to the solution of additional significant special problems with and without rotational symmetry. Finally, we refer to a recent paper by Ana Moisil [10] (1950), in which a formal scheme due to Gr. C. Moisil [11] is used to extend Galerkin's [6] stress-function approach to media with transverse isotropy. The method of derivation employed here does not assure the completeness of the approach which

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[3]See [1], p. 274.

[4]This observation, according to Mindlin [5], is due to H. M. Westergaard.

involves three stress functions, each satisfying a partial differential equation of the sixth order. No attempt is made to effect a reduction in the unnecessarily high order of the equations characterizing the stress functions, which is an essential disadvantage from the point of view of applications, and the case of axisymmetric stress fields is not considered especially.

It is the purpose of the present paper to present a systematic derivation of, and to supply the apparently still missing completeness proof for, Lekhnitsky's approach to problems with torsionless axisymmetry in a medium possessing transverse isotropy. To this end we first generalize a theorem of Almansi [12] concerning the representation of the general solution of the equation $\nabla^{2n} F = 0$ in terms of harmonic functions. The theorem established here may be of interest beyond the present application. The material in Sections 3, 4, although primarily expository, has briefly been included in order to render the present paper sensibly self-contained.

2. A Generalization of Almansi's Theorem

In this section we prove the following theorem:

Let R be a region of the (ρ,z)-plane such that a straight line parallel to the z-axis intersects the boundary of R in at most two points. Let $F_n(\rho,z)$ be a solution of

$$\prod_{i=1}^{n} \nabla_i^2 F_n \equiv \nabla_1^2\nabla_2^2 \cdots \nabla_{n-1}^2 \nabla_n^2 F_n = 0 \quad \text{in} \quad R \tag{2.1}$$

where

$$\left.\begin{aligned} \nabla_i^2 &= \Lambda + c_i^2 D_z^2 \\ \Lambda &= \sum_{i=0}^{r} p_i(\rho)\frac{\partial^i}{\partial\rho^i}, \qquad D_z = \frac{\partial}{\partial z}, \end{aligned}\right\} \tag{2.2}$$

the c_i are (not necessarily real) constants, and the functions p_i are continuous in R with $p_r \neq 0$. Then F_n admits the representation,

$$F_n(\rho,z) = F_{n-1}(\rho,z) + z^m \overset{(n)}{F}(\rho,z) \tag{2.3}$$

where,

$$\prod_{i=1}^{n-1} \nabla_i^2 F_{n-1} = 0, \qquad \nabla_n^2 \overset{(n)}{F} = 0, \tag{2.4}$$

and m $(0 \leq m \leq n-1)$ *is the number of the coefficients* c_i^2 $(i = 1, 2, \ldots, n-1)$ *which are equal to* c_n^2.

For future reference, we first note the trivial identities

$$\nabla_i^2\left[A(z)B(\rho,z)\right] = A\nabla_i^2 B + c_i^2(A_{,zz}B + 2A_{,z}B_{,z}), \tag{2.5}$$ [5]

$$\nabla_j^2 \overset{(i)}{F} = (c_j^2 - c_i^2)\overset{(i)}{F}_{,zz} \quad \text{if} \quad \nabla_i^2 \overset{(i)}{F} = 0, \tag{2.6}$$

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[5] Subscripts preceded by a comma denote partial differentiation with respect to the argument indicated.

and observe that the product of any two among the operators $D_z, \Lambda, \nabla_i^2, \nabla_j^2$ is commutative.

As a further preliminary, we show that if $F(\rho, z)$ satisfies $\nabla_i^2 F = 0$ and $m > 0$ is an integer, then

$$\nabla_i^{2m}(z^k F) = \left\{ \begin{array}{ll} 0 & \text{for } k = m-1 \\ 2^m \underline{|m}\, c_i^{2m} D_z^m F & \text{for } k = m \end{array} \right\} \quad (2.7)$$

Let $k = m - 1$. Clearly, (2.7) holds for $m = 1$. We proceed by induction. Thus, assume

$$\nabla_i^{2(m-1)}(z^{m-2} F) = 0.$$

Then,

$$\begin{aligned} \nabla_i^{2m}(z^{m-1}F) &= \nabla_i^{2(m-1)} \nabla_i^2 (z^{m-1}F) \\ &= (m-1)c_i^2 \nabla_i^{2(m-1)} \left[(m-2)z^{m-3}F + 2z^{m-2}F_{,z} \right] = 0, \end{aligned}$$

by hypothesis and (2.5). Next, let $k = m$. Again, (2.7) evidently holds for $m = 1$. Using induction once more, assume

$$\nabla_i^{2(m-1)}(z^{m-1}F) = 2^{m-1} \underline{|m-1}\, c_i^{2(m-1)} D_z^{m-1} F.$$

Then,

$$\begin{aligned} \nabla_i^{2m}(z^m F) &= \nabla_i^{2(m-1)} \nabla_i^2 (z^m F) \\ &= m c_i^2 \nabla_i^{2(m-1)} \left[(m-1)z^{m-2} F + 2z^{m-1} F_{,z} \right] \\ &= 2^m \underline{|m}\, c_i^{2m} D_z^m F, \end{aligned}$$

by hypothesis, (2.5), and the result for $k = m - 1$. This completes the proof of (2.7).

Turning to the proof of the theorem, we need to show that there exists a function $\overset{(n)}{F}(\rho,z)$ which satisfies

$$\nabla_n^2 F^{(n)} = 0, \tag{2.8}$$

$$\prod_{i=1}^{n-1} \nabla_i^2 \left[F_n - z^m F^{(n)}\right] = 0. \tag{2.9}$$

Moreover, we may assume without loss in generality that

$$\left.\begin{array}{ll} c_i^2 = c_n^2 & (i = n - m,\ n - m + 1,\ \ldots,\ n - 1), \\ c_i^2 \neq c_n^2 & (i = 1,\ 2,\ \ldots,\ n - m - 1). \end{array}\right\} \tag{2.10}$$

In view of (2.10) we may write (2.9) as

$$\prod_{i=1}^{n-1} \nabla_i^2 F_n = \prod_{i=1}^{n-m-1} \nabla_i^2 \left\{\nabla_n^{2m}\left[z^m F^{(n)}\right]\right\}, \tag{2.11}$$

and (2.11), by virtue of (2.6), (2.7), and (2.8), becomes,

$$\prod_{i=1}^{n-1} \nabla_i^2 F_n = 2^m \underline{\lfloor m}\, c_n^{2m} D_z^{2n-m-2} F^{(n)} \prod_{i=1}^{n-m-1} (c_i^2 - c_n^2). \tag{2.12}$$

Thus, it suffices to construct a function $\overset{(n)}{F}(\rho,z)$ which satisfies (2.8) and (2.12). To this end let

$$g(\rho,z) = \frac{D_z^{-2n+m+2} \prod_{i=1}^{n-1} \nabla_i^2 F_n}{2^m \underline{\lfloor m}\, c_n^{2m} \prod_{i=1}^{n-m-1} (c_i^2 - c_n^2)}, \tag{2.13}$$

the operator D_z^{-1} being defined by

$$D_z^{-1} G(\rho,z) = \int_{z_0}^{z} G(\rho,\zeta)\, d\zeta, \tag{2.14}$$

where z_0 is the value of z belonging to one of the points of intersection of a straight line parallel to the z-axis with the boundary of R. Evidently, $F^{(n)} = g$ satisfies (2.12). Furthermore,

$$D_z^{2n-m-2} \nabla_n^2 g = 0 \tag{2.15}$$

because of (2.1). Hence,

$$\nabla_n^2 g = \sum_{k=0}^{2n-m-3} z^k f_k(\rho). \tag{2.16}$$

Now, let $g_*(\rho, z)$ be defined by

$$\left.\begin{aligned} g_* &= \sum_{k=0}^{2n-m-3} z^k h_k(\rho), \\ \Lambda h_{2n-m-3} &= f_{2n-m-3}, \qquad \Lambda h_{2n-m-4} = f_{2n-m-4} \\ \Lambda h_k &= f_k - c_n^2 (k+1)(k+2) h_{k+2} \quad (k = 0,1,\ldots,2n-m-5). \end{aligned}\right\} \tag{2.17}$$

The function g_* so determined has the properties

$$\nabla_n^2 g_* = \nabla_n^2 g, \qquad D_z^{2n-m-2} g_* = 0. \tag{2.18}$$

It follows from (2.13), (2.18) that

$$F^{(n)} = g - g_* \tag{2.19}$$

satisfies both (2.8) and (2.12). This completes the proof of the theorem.

Successive applications of the foregoing theorem at once yield a representation of any solution of

$$\prod_{i=1}^{n} \nabla_i^2 F = 0 \tag{2.20}$$

in terms of solutions $F^{(i)}$ of the equations

$$\nabla_i^2 F^{(i)} = 0 \qquad (i = 1, 2, \ldots, n). \tag{2.21}$$

In particular, if all c_k^2 $(i = 1, 2, \ldots, n)$ are distinct,

$$F = \sum_{k=1}^{n} F^{(k)}, \tag{2.22}$$

whereas if all c_k^2 coalesce,

$$F = \sum_{k=1}^{n} z^k F^{(k)}. \tag{2.23}$$

In connection with the plane problem of elasticity theory for an orthotropic medium, special interest is attached to the case in which (2.2) assume the form,

$$\Lambda = \frac{\partial^2}{\partial \rho^2}, \qquad \nabla_i^2 = \frac{\partial^2}{\partial \rho^2} + c_i^2 \frac{\partial^2}{\partial z^2}. \tag{2.24}$$

Here ρ and z are Cartesian coordinates, and ∇_i^2 is a modified two-dimensional Laplacian operator. In the rotationally symmetric problem appropriate to a medium with transverse isotropy, on the other hand, we encounter the case in which

$$\Lambda = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}, \qquad \nabla_i^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + c_i^2 \frac{\partial^2}{\partial z^2}, \tag{2.25}$$

where ρ and z are circular cylindrical coordinates and ∇_i^2 is a modified axisymmetric Laplacian operator.

Finally, if ∇_i^2 in (2.20) is defined as in (2.24) or (2.25), and $c_i^2 = 1$ $(i = 1, 2, \ldots, n)$, then (2.23) becomes identical with Almansi's [12] representation of the general solution of $\nabla^{2n} F = 0$ in terms of harmonic functions, for the case in which ∇^2 is the two-dimensional or the axisymmetric three-dimensional Laplacian operator. This special instance arises in the well known applications of Almansi's theorem to the plane and to the rotationally symmetric problem in the classical theory of elasticity of isotropic media.

3. Stress-Strain Relations and Strain-Energy Function for a Medium with Transverse Isotropy

Let us, temporarily, adopt the notation,

$$\left.\begin{aligned} &\tau_1 = \tau_{xx},\ \tau_2 = \tau_{yy},\ \tau_3 = \tau_{zz},\ \tau_4 = \tau_{yz},\ \tau_5 = \tau_{zx},\ \tau_6 = \tau_{xy}; \\ &e_1 = e_{xx},\ e_2 = e_{yy},\ e_3 = e_{zz},\ e_4 = e_{yz},\ e_5 = e_{zx},\ e_6 = e_{xy}. \end{aligned}\right\} \quad (3.1)$$

Here, $\tau_{xx}, \ldots$ and $e_{xx}, \ldots$ are the Cartesian components of stress and "infinitesimal" strain, the strains being defined by

$$e_{xx} = u_{x,x}, \ \ldots. \quad e_{yz} = u_{y,z} + u_{z,y}, \ \ldots. \quad (3.2)$$

where $[u_x, u_y, u_z]$ are the Cartesian scalar components of the displacement vector.

The general linear stress-strain law now assumes the form,

$$\tau_i = c_{ij}\, e_j, \quad (3.3)$$

in which i, j range over the integers one to six, and the usual summation convention is employed. For a homogeneous medium the c_{ij} are constants.[6]

A necessary and sufficient condition for the existence of a strain-energy function $W(e_1, \ldots, e_6)$, such that

$$\tau_i = \frac{\partial W}{\partial e_i}, \quad (3.4)$$

is that

$$c_{ji} = c_{ij}. \quad (3.5)$$

Moreover,

$$W = \frac{1}{2} c_{ij}\, e_i\, e_j. \quad (3.6)$$

---

[6] Note that the present definition of the elastic constants c_{ij} coincides with that adopted by Love [1], p. 99.

We now impose the condition (transverse isotropy) that the stress-strain law (3.3), (3.5) be invariant under an arbitrary rotation about the z-axis. This is the case if and only if

$$\left.\begin{aligned} &c_{ij} = 0 \quad (i = 1,2,3;\ j = 4,5,6),\ (i = 4,5,6;\ j \neq i) \\ &c_{11} = c_{22},\ c_{13} = c_{23},\ c_{44} = c_{55},\ 2c_{66} = c_{11} - c_{12}. \end{aligned}\right\} \quad (3.7)$$

With the notation,

$$\left.\begin{aligned} &c_{11} = a,\ c_{33} = \bar{a},\ c_{44} = \mu,\ c_{66} = \bar{\mu} \\ &c_{12} = a - 2\bar{\mu},\ c_{13} = b, \end{aligned}\right\} \quad (3.8)$$

the stress-strain law (3.3), (3.5), subject to (3.7), (3.8), becomes

$$\left.\begin{aligned} &\tau_{xx} = ae_{xx} + (a - 2\bar{\mu})\, e_{yy} + be_{zz} \\ &\tau_{yy} = ae_{yy} + (a - 2\bar{\mu})\, e_{xx} + be_{zz} \\ &\tau_{zz} = b(e_{xx} + e_{yy}) + \bar{a}e_{zz} \\ &\tau_{yz} = \mu e_{yz}, \qquad \tau_{zx} = \mu e_{zx}, \qquad \tau_{xy} = \bar{\mu}\, e_{xy}, \end{aligned}\right\} \quad (3.9)$$

whereas (3.6) appears as

$$\begin{aligned} 2W = a(e_{xx}^2 + e_{yy}^2) + \bar{a}e_{zz}^2 + 2(a - 2\bar{\mu})e_{xx}e_{yy} \\ + 2b(e_{xx} + e_{yy})e_{zz} + \mu(e_{yz}^2 + e_{zx}^2) + \bar{\mu}\, e_{xy}^2. \end{aligned} \quad (3.10)$$

The stress-strain relations (3.9) involve five independent elastic constants, of which μ and $\bar{\mu}$ have an obvious physical meaning.

Inverting the first three of (3.9), we reach

$$
\left.
\begin{aligned}
e_{xx} &= \frac{1}{E}\left[\tau_{xx} - \nu\tau_{yy} - \bar{\nu}\tau_{zz}\right] \\
e_{yy} &= \frac{1}{E}\left[\tau_{yy} - \nu\tau_{xx} - \bar{\nu}\tau_{zz}\right] \\
e_{zz} &= \frac{1}{\bar{E}}\left[\tau_{zz} - \frac{\bar{\nu}\bar{E}}{E}\left(\tau_{xx} + \tau_{yy}\right)\right],
\end{aligned}
\right\} \quad (3.11)
$$

where

$$
\left.
\begin{aligned}
E &= \frac{4\bar{\mu}(a\bar{a} - b^2 - \bar{a}\bar{\mu})}{a\bar{a} - b^2}, \qquad & \bar{E} &= \frac{a\bar{a} - b^2 - \bar{a}\bar{\mu}}{a - \bar{\mu}} \\
\nu &= \frac{a\bar{a} - b^2 - 2\bar{a}\bar{\mu}}{a\bar{a} - b^2}, \qquad & \bar{\nu} &= \frac{2b\bar{\mu}}{a\bar{a} - b^2}.
\end{aligned}
\right\} \quad (3.12)
$$

Furthermore,

$$
E = 2\bar{\mu}(1 + \nu). \tag{3.13}
$$

The physical significance of the four new elastic constants defined in (3.12) is immediate from (3.11).

A trivial computation, based on (3.10), yields the following necessary and sufficient conditions for the positive definiteness of W:

$$
a > 0,\ \bar{a} > 0,\ \mu > 0,\ \bar{\mu} > 0,\ a\bar{a} - b^2 - \bar{a}\bar{\mu} > 0, \tag{3.14}
$$

or, equivalently,

$$
E > 0,\ \bar{E} > 0,\ \mu > 0,\ \bar{\mu} > 0,\ -1 < \nu < 1,\ 1 - \nu > 2\,\frac{E\bar{\nu}^2}{\bar{E}}. \tag{3.15}
$$

In the special case of isotropy, we have

$$
\left.
\begin{aligned}
a = \bar{a} &= 2\mu + \lambda = \frac{2(1 - \nu)\mu}{1 - 2\nu}, \qquad b = \lambda = \frac{2\mu\nu}{1 - 2\nu}, \\
\mu &= \bar{\mu} = \frac{E}{2(1 + \nu)}.
\end{aligned}
\right\} \quad (3.16)
$$

4. Basic Equations for a Medium with Transverse Isotropy in the Case of Torsionless Axisymmetry

With reference to cylindrical coordinates (ρ, Θ, γ), where

$$x = \rho \cos \gamma, \qquad y = \rho \sin \gamma, \qquad z = z, \tag{4.1}$$

the displacement field in the presence of torsionless axisymmetry about the z-axis, is characterized by

$$u_\rho = u_\rho(\rho, z), \qquad u_\gamma = 0, \qquad u_z = u_z(\rho, z). \tag{4.2}$$

The associated field of strain now becomes

$$\left.\begin{aligned} &e_{\rho\rho} = u_{\rho,\rho}, \qquad e_{\gamma\gamma} = \frac{u_\rho}{\rho}, \qquad e_{zz} = u_{z,z}, \\ &e_{\rho z} = u_{\rho,z} + u_{z,\rho}, \qquad e_{\gamma\rho} = e_{z\gamma} = 0, \end{aligned}\right\} \tag{4.3}$$

and, according to (3.9), the stress-strain relations are

$$\left.\begin{aligned} \tau_{\rho\rho} &= a e_{\rho\rho} + (a - 2\bar{\mu}) e_{\gamma\gamma} + b e_{zz} \\ \tau_{\gamma\gamma} &= a e_{\gamma\gamma} + (a - 2\bar{\mu}) e_{\rho\rho} + b e_{zz} \\ \tau_{zz} &= \bar{a} e_{zz} + b(e_{\rho\rho} + e_{\gamma\gamma}) \\ \tau_{\rho z} &= \mu e_{\rho z}, \qquad \tau_{\gamma\rho} = \tau_{\gamma z} = 0. \end{aligned}\right\} \tag{4.4}$$

The stress equations of equilibrium, in the absence of body forces, by virtue of (4.2), (4.3), and (4.4) here reduce to

$$\left.\begin{aligned} &\tau_{\rho\rho,\rho} + \tau_{\rho z,z} + \frac{\tau_{\rho\rho} - \tau_{\gamma\gamma}}{\rho} = 0 \\ &\tau_{\rho z,\rho} + \tau_{zz,z} + \frac{\tau_{\rho z}}{\rho} = 0 \end{aligned}\right\} \tag{4.5}$$

and substitution of (4.2), (4.3), (4.4) into (4.5) yields

$$\left.\begin{aligned} ae_{,\rho} + \mu u_{\rho,zz} + (\mu + b - a)u_{z,\rho z} &= 0, \\ (\mu + b)e_{,z} + \frac{\mu}{\rho}\frac{\partial}{\partial\rho}(\rho u_{z,\rho}) - (\mu + b - \bar{a})u_{z,zz} &= 0, \end{aligned}\right\} \quad (4.6)$$

in which

$$e = e_{\rho\rho} + e_{\gamma\gamma} + e_{zz} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho u_{\rho}) + u_{z,z}. \quad (4.7)$$

5. Derivation and Completeness of Lekhnitsky's Stress-Function Approach

Suppose $u_\rho(\rho,z)$ and $u_z(\rho,z)$ satisfy the equilibrium equations (4.6) in a region R of the meridional half-plane $\rho \geq 0$, $-\infty < z < \infty$, and let R have the property that straight lines parallel to the coordinate axes intersect the boundary of R in at most two points. We may then define two functions $U(\rho,z)$ and $V(\rho,z)$ by means of

$$u_\rho = U_{,\rho}, \qquad u_z = V_{,z} \tag{5.1}$$

For convenience, we introduce the operators

$$\left.\begin{aligned} \nabla_*^2 &= \nabla^2 - D_z^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right), \\ D_z &= \frac{\partial}{\partial z}, \end{aligned}\right\} \tag{5.2}$$

∇^2 being the axisymmetric Laplacian. Substitution of (5.1) in (4.6), with the aid of (4.7) and (5.2), yields

$$\left.\begin{aligned} &\frac{\partial}{\partial\rho}\left[a\nabla_*^2 U + \mu D_z^2 U + (\mu + b)\, D_z^2 V\right] = 0 \\ &\frac{\partial}{\partial z}\left[(\mu + b)\, \nabla_*^2 U + \mu\nabla_*^2 V + \bar{a}D_z^2 V\right] = 0, \end{aligned}\right\} \tag{5.3}$$

or, since (5.1) determine U and V only within arbitrary additive functions of z and ρ, respectively,

$$\left.\begin{aligned} &a\nabla_*^2 U + \mu D_z^2 U + (\mu + b)\, D_z^2 V = 0 \\ &(\mu + b)\, \nabla_*^2 U + \mu\nabla_*^2 V + \bar{a}D_z^2 V = 0 \end{aligned}\right\} \tag{5.4}$$

On eliminating first V and then U between (5.4), we reach

$$\Omega U = 0, \qquad \Omega V = 0, \tag{5.5}$$

the operator Ω being given by

$$\Omega = \nabla_*^4 + \frac{a\bar{a} - 2\mu b - b^2}{a\mu} \nabla_*^2 D_z^2 + \frac{\bar{a}}{a} D_z^4. \tag{5.6}$$

Consistent with the notation introduced in (2.2), we now define the operator ∇_i^2 through

$$\nabla_i^2 = \nabla_*^2 + c_i^2 D_z^2 \quad (i = 1, 2) \tag{5.7}$$

and seek to determine the coefficients c_1^2, c_2^2 in such a way that

$$\Omega = \nabla_1^2 \nabla_2^2. \tag{5.8}$$

To this end, we expand the right-hand member of (5.8) as a polynomial in ∇_*^2, D_z^2 and equate its coefficients to the corresponding coefficients appearing in (5.6). This leads to

$$c_1^2 + c_2^2 = \frac{a\bar{a} - 2b\mu - b^2}{a\mu}, \quad c_1^2 c_2^2 = \frac{\bar{a}}{a}, \tag{5.9}$$

so that c_1^2, c_2^2 are the roots of the quadratic,

$$Q(\xi) = \xi^2 + \frac{b^2 + 2b\mu - a\bar{a}}{a\mu} \xi + \frac{\bar{a}}{a} = 0. \tag{5.10}$$

The inequalities (3.14) imply that $Q(\xi)$ cannot have a negative or vanishing root. Conditions (3.14), however, do not preclude the existence of complex roots. Indeed, c_1^2 and c_2^2 are real and distinct, coalescent, or conjugate complex, according as

$$a\bar{a} - (2\mu + b)^2 \gtreqless 0. \tag{5.11}$$

In the special instance of isotropy we conclude from (3.16), (5.10) that $c_1^2 = c_2^2 = 1$, whence $\nabla_i^2 = \nabla^2$ $(i = 1, 2)$ and $\Omega = \nabla^4$.

In view of (5.5), (5.8), and by virtue of the theorem proved in Section 2, U admits the representation

$$\left.\begin{aligned} U &= \overset{(1)}{B} + z^k \overset{(2)}{B} \\ \nabla_1^2 \overset{(1)}{B} &= 0, \qquad \nabla_2^2 \overset{(2)}{B} = 0 \end{aligned}\right\} \tag{5.12}$$

where

$$\left. k = \begin{cases} 0 & \text{if } c_1^2 \neq c_2^2 \\ 1 & \text{if } c_1^2 = c_2^2. \end{cases} \right\} \tag{5.13}$$

It is expedient to define two new functions $\overset{(1)}{A}(\rho,z)$, $\overset{(2)}{A}(\rho,z)$ by means of

$$\left.\begin{aligned} \overset{(1)}{B} &= \overset{(1)}{A}_{,z} + k\overset{(2)}{A}_{,z}\,, \qquad \overset{(2)}{B} = \overset{(2)}{A}_{,zz} \\ \nabla_1^2 \overset{(1)}{A} &= 0, \qquad \nabla_2^2 \overset{(2)}{A} = 0. \end{aligned}\right\} \tag{5.14}$$ [7]

From (5.14), (5.12), (5.13), (5.7) follows

$$U = D_z \left[\overset{(1)}{A} + z^k \overset{(2)}{A}_{,z} \right] \tag{5.15}$$

$$\nabla_*^2 U = - D_z^3 \left[c_1^2 \overset{(1)}{A} + c_2^2 z^k \overset{(2)}{A}_{,z} - 2kc_2^2 \overset{(2)}{A} \right] \tag{5.16}$$

Substituting (5.15), (5.16) in the first of (5.4), and performing two successive z-integrations, we obtain

$$\begin{aligned} (\mu + b)\, V = D_z \Big[(ac_1^2 - \mu)\overset{(1)}{A} &+ (ac_2^2 - \mu) z^k \overset{(2)}{A}_{,z} \\ &- 2ac_2^2 k\overset{(2)}{A} + z\, f_1 + z^2 f_2 \Big], \end{aligned} \tag{5.17}$$

where f_1 and f_2 are functions of ρ alone. We first treat the general case in which the degeneracy $\mu = -b$ is ruled out.

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[7] The existence of $\overset{(1)}{A}$ and $\overset{(2)}{A}$ is readily verified with the aid of the last two of (5.12).

<u>Case 1</u>: $\mu + b \neq 0$

In order to determine the nature of the functions f_1, f_2, we substitute (5.16), (5.17) into the second of (5.4). This lengthy computation, with the aid of (5.10), (5.13), (5.14), leads to

$$\nabla_*^2 f_1 = 0, \qquad \nabla_*^2 f_2 = 0 \tag{5.18}$$

Now define $\Phi(\rho,z)$ by

$$\left.\begin{aligned} \Phi &= -\frac{1}{\mu + b}\left[A^{(1)} + z^k A^{(2)}_{,z} + f_3(\rho)\right], \\ a\nabla_*^2 f_3 &= -2f_2. \end{aligned}\right\} \tag{5.19}$$

From (5.19), (5.15), (5.1.), (5.1; follows

$$\left.\begin{aligned} u_\rho &= -(\mu + b)\,\Phi_{,\rho z} \\ u_z &= a\nabla^2\Phi + (\mu - a)\,\Phi_{,zz}, \end{aligned}\right\} \tag{5.20}$$

and (5.13), (5.14), (5.18), (5.19) imply

$$\Omega\Phi \equiv \nabla_1^2\,\nabla_2^2\,\Phi = 0. \tag{5.21}$$

We have shown that, barring the exceptional case $\mu + b = 0$, every solution of the displacement equations of equilibrium (4.6) admits the representation (5.20) in terms of the stress function Φ which is subject to (5.21). Moreover, a direct substitution confirms that every displacement field of the form (5.20), with Φ a particular solution of (5.21), satisfies (4.6) even if $\mu + b = 0$. Equations (5.20), (5.21), except for an unessential constant multiplier of Φ, are identical with the stress-function approach given by Lekhnitsky [3], and independently arrived at by Elliot [7]. Applying the theorem of Section 2 to (5.21),

and recalling (5.7), we note that the general solution of (5.21) may be written as

$$\Phi(\rho,z) = \Phi_1(\rho, z/c_1) + z^k \Phi_2(\rho,z/c_2), \tag{5.22}$$

where $\Phi_1(\rho,z)$, $\Phi_2(\rho,z)$ are arbitrary axisymmetric harmonic functions and k is given by (5.13). The cylindrical components of stress belonging to the displacements (5.20) are obtained with the aid of (4.3), (4.4) and may be omitted here.

In particular, if the medium is isotropic, let

$$\chi = \frac{2\mu^2}{1-2\nu}\Phi. \tag{5.23}$$

In view of (3.16), Equations (5.20), (5.21) here reduce to

$$\left.\begin{aligned} 2\mu u_\rho = -\chi_{,\rho z}, \quad 2\mu u_z = 2(1-\nu)\nabla^2\chi - \chi_{,zz}, \\ \nabla^4\chi = 0, \end{aligned}\right\} \tag{5.24}$$

which is Love's stress-function approach[8] to isotropic problems with torsionless axisymmetry.

<u>Case 2</u>: $\mu + b = 0$.

In this special instance (5.4) degenerate into

$$\nabla_*^2 U + \frac{\mu}{a} U_{,zz} = 0, \qquad \nabla_*^2 V + \frac{\bar{a}}{\mu} V_{,zz} = 0, \tag{5.25}$$

and by (5.10),

$$c_1^2 = \frac{\mu}{a}, \qquad c_2^2 = \frac{\bar{a}}{\mu}. \tag{5.26}$$

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[8] See [1], p. 276.

Consequently, according to (5.1), (5.7), the general solution of (4.6) now appears as

$$\left.\begin{aligned} u_\rho = U_{,\rho}, \qquad & u_z = V_{,z} \\ \nabla_1^2 U = 0, \qquad & \nabla_2^2 V = 0 \end{aligned}\right\} \tag{5.27}$$

and is thus again expressible in terms of two arbitrary harmonic functions.

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